Primitive School on the MCG

Curves & the Alexander Method

12/03 - 14/03
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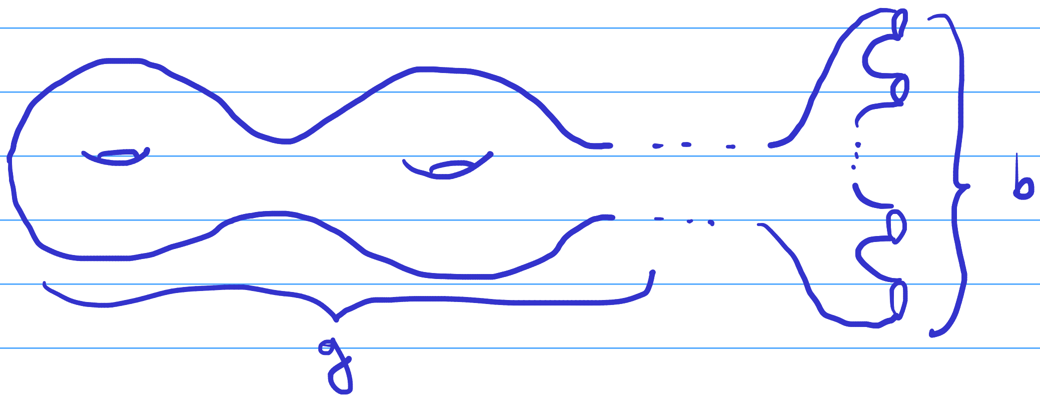
§ 1. Mod(S)

A **surface** is a compact 2-dimensional manifold (possibly with boundary).

Theorem (Möbius - Radó) Up to homeomorphism, a surface is determined by its genus g & the number b of boundary components of each connected component.

$S_{g,b}$ =



Remark If we remove m points from $S_{g,b}$, we obtain the (non-compact) two-manifold $S_{g,b,m}$. We will also call them surfaces. We define the **Euler characteristic**

$$\chi(S_{g,b,m}) = 2 - 2g - b - m.$$

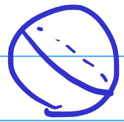
Any 3 of the 4 numbers $g, b, m, \chi(S)$ determine S up to homeomorphism. If S is disconnected, $\chi(S)$ is the sum of the Euler characteristics of its components.

Observe that any surface admits a differential structure, & hence a metric. We assume S is connected until section 3.

Theorem If $\chi(S) < 0$ (resp $\chi(S) = 0$) then S admits a hyperbolic (resp flat) metric with totally geodesic boundary

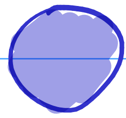
Example There are only 7 surfaces with $\chi(S) \geq 0$:

$\chi(S) = 2$



$S_{0,0,0}$

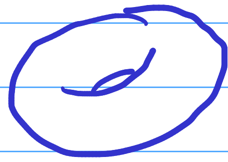
$\chi(S) = 1$



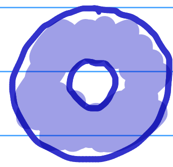
$S_{0,1,0}$

$S_{0,0,1}$

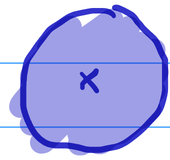
$\chi(S) = 0$



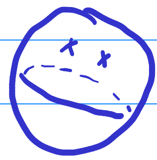
$S_{1,0,0}$



$S_{0,2,0}$



$S_{0,1,1}$



$S_{0,0,2}$

∎

Let

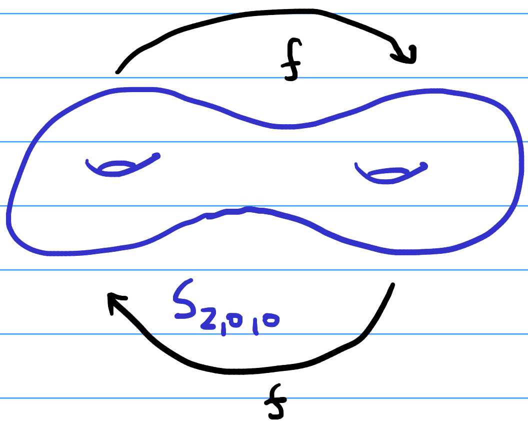
$$\mathrm{Homeo}^+(S, \partial S) = \left\{ f : S \to S \; : \; \begin{array}{l} \text{homeomorphism} \\ \text{orientation preserving} \\ \text{identity on } \partial S \end{array} \right\}$$

Two elements $f, g \in \mathrm{Homeo}^+(S, \partial S)$ are homotopic if $\exists$ $H : S \times [0,1] \to S$ continuous such that $H(-,0) = f$, $H(-,1) = g$ & $H(-,t)$ is the identity on ∂S $\forall t \in [0,1]$. The mapping class group of S is the group

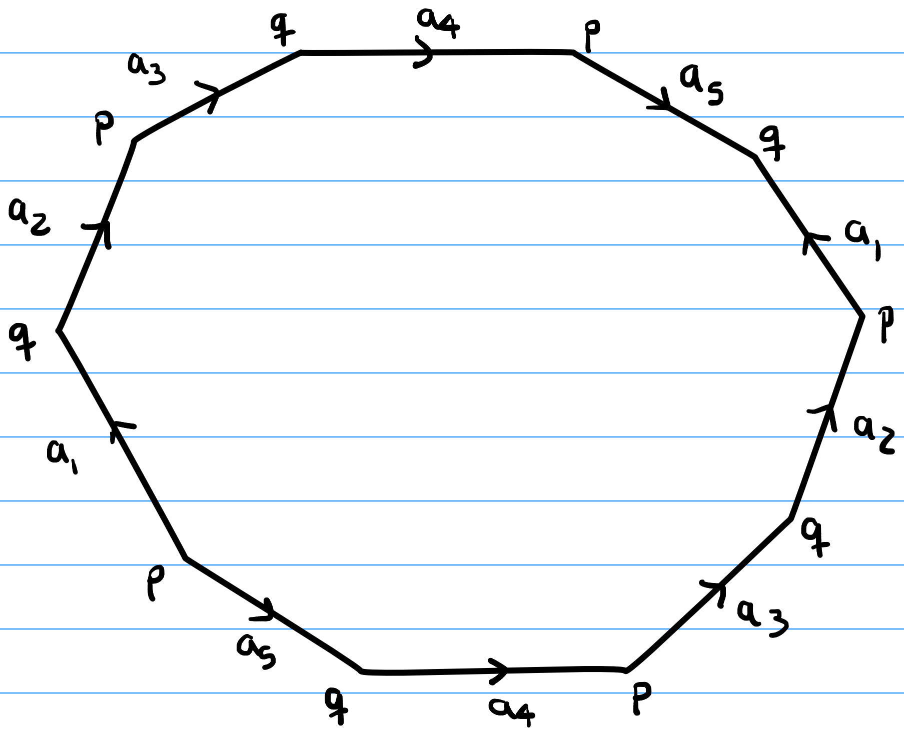
$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S) \, / \, \text{homotopy}.$$

We call its elements mapping classes.

Example Let $S = S_{2,0,0}$. Consider the two mapping classes $[f]$ & $[g]$ defined as follows. The map f is an involution:



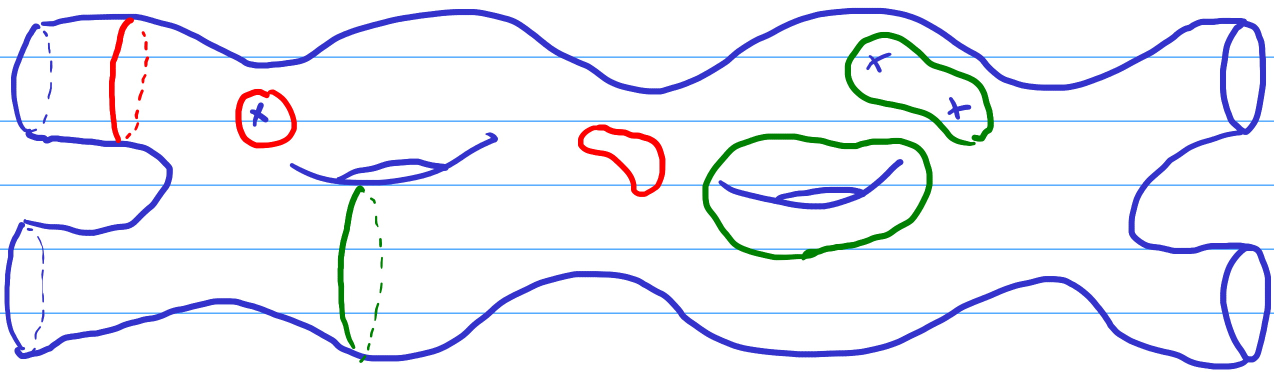
Hence $[f]^2 = 1$. For the map g, represent $S_{2,0,0}$ as a CW complex with two 0-cells p & q, five 1-cells $a_1, \dots, a_5$ and the following one 2-cell:



Let g be the rotation of this 2-cell by $\pi/5$. Hence, $[g]^{10} = 1$. What are the orders of $[f]$ & $[g]$ in $\mathrm{Mod}(S)$? ■

To see if a mapping class is trivial or not, we will study is action on curves.

A simple closed curve (s.c.c.) is an embedded $S^1 \hookrightarrow S$. It is essential if it does not bound a disc, a 1-punctured disc or is parallel to a boundary component.



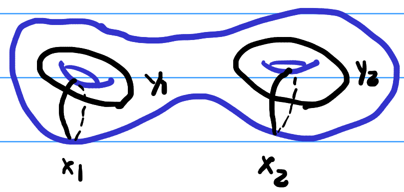
■ Not essential ■ essential

Two s.c.c. α & β are homotopic if $\exists H: S^1 \times [0,1] \to S$ continuous such that $H(-,0) = \alpha$ & $H(-,1) = \beta$. We say α & β are homologous if there is a continuous map $f: S_{g,2,m} \to S$ such that f maps the boundary circles to α & β. Since $S^1 \times [0,1] = S_{0,2,0}$, homotopic implies homologous. Denote the homotopy class of α by (α), and the homology class by $[\alpha] \in H_1(S, \mathbb{Z})$.

Exercise If $[f] = [g] \in \mathrm{Mod}(S)$, $(\alpha) = (\beta)$ and $[\alpha] = [\gamma]$, then $(f\alpha) = (g\beta)$ & $[f\alpha] = [g\gamma]$, so that $\mathrm{Mod}(S)$ acts on both homotopy & homology of s.c.c.

Let $H_1: S \times [0,1] \to S$ be an homotopy from f to g & $H_2: S^1 \times [0,1] \to S$ an homotopy from α to β. Then $H(x,t) = H_1(H_2(x,t), t)$ is an homotopy from $f\alpha$ to $g\beta$. Given $h: S_{g,2,m} \to S$ that sends the boundary to α & γ, $f \circ h$ shows that $[f\alpha] = [f\gamma]$. Now $H_1 \circ \gamma: S^1 \times [0,1] \to S$ shows that $[f\gamma] = [g\gamma]$.

Example Let $S = S_{2,0,0}$ and $[f]$ & $[g]$ be from the previous example. Using cellular homology, one readily checks that the following curves generate mod 2 homology:



& $\{a_1 + a_2, a_1 + a_3, a_1 + a_4, a_1 + a_5\}$

In the first basis, f acts as the matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

and in the second basis g acts as the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

so that both are non-trivial. Those matrices have order 2 & 5, and $[g]^5$ acts as -1 on $H_1(S, \mathbb{Z})$, so that $[f]$ has order 2 & $[g]$ has order 10. $\blacksquare$

Addenda I) Two homeomorphisms $f, g: S \to S$ are isotopic if $\exists\, H: S \times [0,1] \to S$ continuous s.t. $H(-,0) = f$, $H(-,1) = g$ and $H(-,t)$ is a homeomorphism $\forall t \in [0,1]$. Two s.c.c. $\alpha, \beta: S^1 \to S$ are isotopic if $\exists\, H: S \times [0,1] \to S$ continuous s.t. $H(-,0) = \alpha$, $H(-,1) = \beta$ and $H(-,t)$ is an embedding $\forall t \in [0,1]$.

Theorem (Baer, '20s) $(\alpha) = (\beta)$ iff they are isotopic. If f and g are orientation preserving, then $[f] = [g]$ iff they are isotopic.

II) If $f: S \to S$ is an orientation preserving homeomorphism, then f is homotopic to a diffeomorphism (Munkres, Smale, Whitehead, 1950's). If $[f] = [g]$ are diffeomorphisms, they are isotopic by a smooth isotopy. Hence:

$$\begin{aligned} \mathrm{Mod}(S) &:= \mathrm{Homeo}^+(S, \partial S) \,/\, \text{homotopy} \\ &= \pi_0(\mathrm{Homeo}^+(S, \partial S)) \\ &\cong \mathrm{Diff}^+(S, \partial S) / \text{homotopy} \\ &= \pi_0(\mathrm{Diff}^+(S, \partial S)) \end{aligned}$$

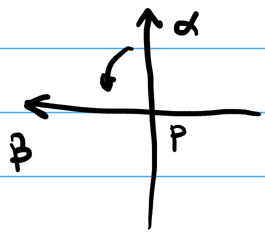
III) Any essential s.c.c. is homotopic to a smooth one (true by a straightening argument in any compact Riemannian manifold). If α & β are smooth and $(\alpha) = (\beta)$, there is a smooth homotopy between them. I couldn't find a reference for whether there is a smooth isotopy or not.

§2 Intersection numbers

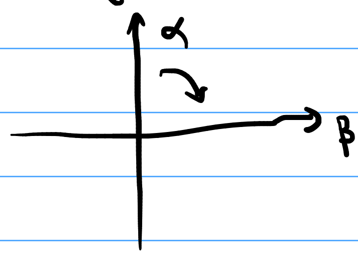
Two smooth s.c.c. α & β are transverse if $T_p\alpha + T_p\beta = T_pS$ for every $p \in \alpha \cap \beta$. A smooth map $h: \tilde{S} \to S$ is transverse to β if $\mathrm{Im}(dh_q) + T_{hq}\beta = T_{hq}S$ for every $q \in h^{-1}\beta$.

Let α & β be transverse, oriented s.c.c. and $p \in \alpha \cap \beta$. We say p is:

positive



negative



Define the algebraic intersection number of α & β to be:

$$\hat{i}(\alpha, \beta) = \# \text{positive intersections} - \# \text{negative intersections}$$

Clearly $\hat{i}(\alpha,\beta) = -\hat{i}(\beta,\alpha)$.

(There are finitely many intersections by Thom's intersection theorem, Bredon Sec. II.7).

Prop. If $[\alpha] = [\alpha']$ & $[\beta] = [\beta']$, then $\hat{i}(\alpha,\beta) = \hat{i}(\alpha',\beta')$.

Sketch We have $H_1(S,\mathbb{Z}) \hookrightarrow H_1(S,\mathbb{R}) \simeq H_c^1(S,\mathbb{R}) \cong H^1_{dR,c}(S,\mathbb{R})$. The dual of α is the 1-form α^* given by the differential of a step function supported on a tubular neighborhood of α. Then:

$$\hat{i}(\alpha,\beta) = \int_\beta \alpha^*,$$

showing that $\hat{i}$ only depends on $[\alpha]$ & $[\beta]$. ∎

Remark By Poincaré duality we know $H_1(S,\mathbb{Z}) \cong H^1_c(S,\mathbb{Z})$ and $H^2_c(S,\mathbb{Z}) \simeq \mathbb{Z}$, the latter being a choice of orientation for S. Then:

$$\hat{i}(\alpha,\beta) = \alpha^* \cup \beta^*.$$

The follow observation is also immediate:

Lemma $\hat{i}(f\alpha, f\beta) = \hat{i}(\alpha,\beta) \quad \forall f \in \mathrm{Diffeo}^+(S)$.

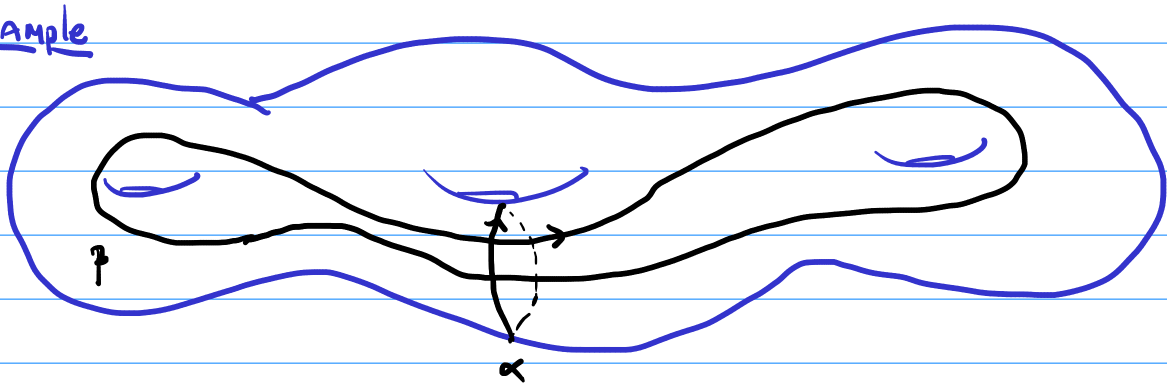
Proof $f\alpha$ & $f\beta$ are also transverse and have positive & negative intersections in bijection with those of f. ∎

The geometric intersection number of α & β is:

$$i(\alpha,\beta) = \min \{ |\alpha' \cap \beta'| : (\alpha') = (\alpha),\ (\beta') = (\beta) \}$$

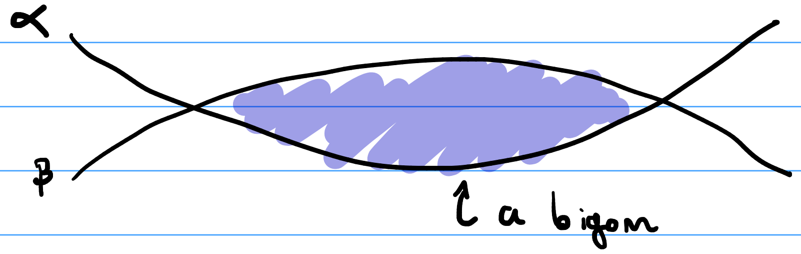
Clearly $i(\alpha,\beta) = i(\beta,\alpha)$ and it depends only on (α) & (β). It is also clear that $i(f\alpha, f\beta) = i(\alpha,\beta)$ $\forall f \in \mathrm{Homeo}^+(S)$. Moreover, $i(\alpha,\beta)$ and $\hat{i}(\alpha,\beta)$ always have the same parity.

Example



We have $\hat{i}(\alpha,\beta) = 0$. What is $i(\alpha,\beta)$?

We say two transverse s.c.c α & β are in minimal position if $|\alpha \cap \beta| = i(\alpha, \beta)$. We say they form a bigon if the region between two consecutive intersections bounds a disk.



Theorem (Bigon theorem) Two s.c.c α & β are in minimal position iff they do not form a bigon.

Corollary If $|\alpha \cap \beta| = 1$, they are in minimal position.

First we need a lemma:

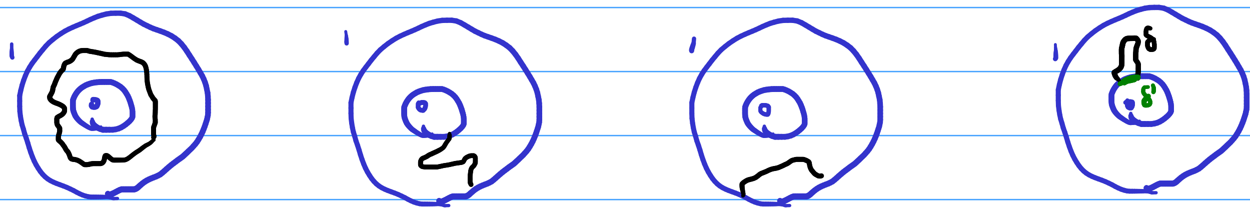
Lemma If α, β s.c.c. do not form bigons, then any lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α & β intersect at most once.

Proof If $\chi(S) > 0$, the lemma is simply the Jordan curve theorem. Hence, we assume $\chi(S) \leq 0$ so that $\tilde{S} \simeq \mathbb{R}^2$. Suppose $\tilde{\alpha}$ & $\tilde{\beta}$ intersect at least twice and let $p: \tilde{S} \to S$ be the covering map. By the Jordan curve theorem, some piece of $\tilde{\alpha}$ and $\tilde{\beta}$ bound a disk D_0.

Define the following graph: its vertices are points in D_0 over $\alpha \cap \beta$, and its edges are paths over α or β. Since p is a covering map & D_0 is compact, this graph is finite, with edges labelled α or β. Moreover, there is at least 1 "disk": two vertices v_1 & v_2 joined by two paths, one labelled only α and the other only β, exactly the boundary of D_0. This disk has no vertices along the α path, but may have vertices on the β path. The lifts of α passing through such vertices are equal or disjoint. Hence, there is a minimal disk: one whose interior does contain lifts of α & β, or equivalently only has 2 vertices on its boundary, call it D.

Write the two vertices of D as w_1, w_2, and its edges as $\tilde{\alpha}_1$ & $\tilde{\beta}_1$. We have $p(w_1) \neq p(w_2)$ as the intersections have different signs. Minimality ensures p is injective on $\partial D = \tilde{\alpha}_1 \cup \tilde{\beta}_1$. If $x, y \in \text{int} D$ are such that $p(x) = p(y)$, let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be the deck transformation sending x to y. If $\phi \neq id$, then $\phi(\partial D) \cap \partial D = \emptyset$. Hence either $\phi(D) \subseteq D$ or $\phi^{-1}(D) \subseteq D$, so by the Brower fixed point theorem ϕ fixes a point, a contradiction. Thus, $p(D)$ is the desired bigon. ∎

Proof of the bigon thm Suppose α & β are not in minimal position & let $H: S^1 \times [0,1] \to S$ be an homotopy of α that reduces the intersection. We may assume H is transverse to β. By Thom's transversality theorem, $H^{-1}(\beta)$ is an embedded 1-submanifold of $S^1 \times [0,1]$:



possible components of $H^{-1}(\beta)$.

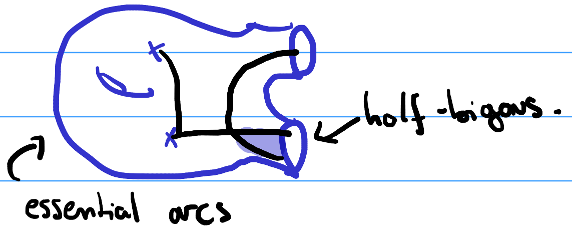
Since H reduces the intersection, the last case must occur. Their path $\delta \cup \delta'$ bounds a disc in $S^1 \times [0,1]$ and hence the corresponding (maybe not simple) closed curve $H(\delta \cup \delta')$ in S is null-homotopic. It lifts in $\tilde{S}$ to a null-homotopic curve containing one arc over α and one over β, so by the lemma α & β form a bigon. ∎

Hence, to get two curves in minimal position, just remove all bigons one by one.

Corollary If $\chi(S) < 0$, distinct simple closed geodesics are in minimal position.

(geodesics in $\mathbb{H}^2$ are unique)

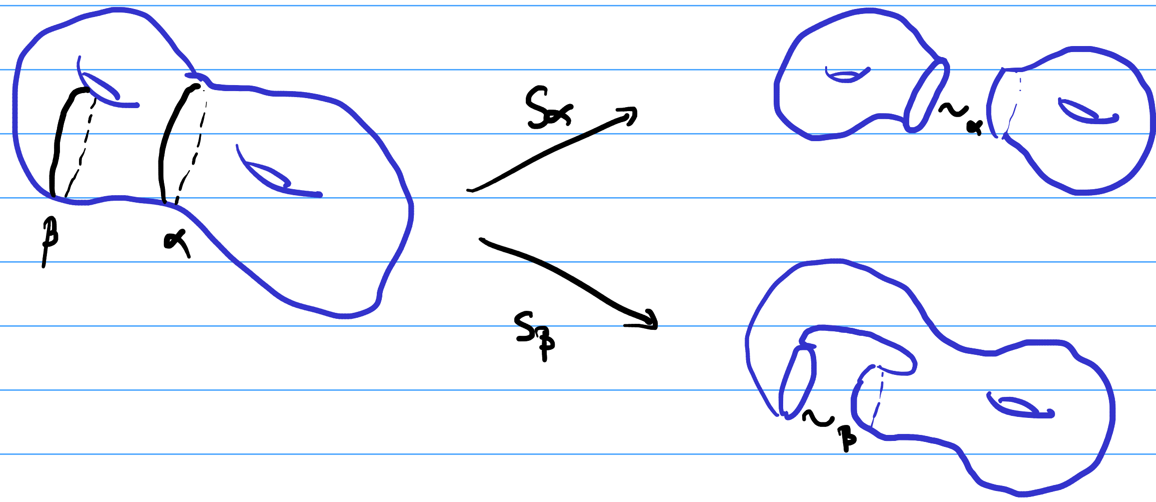
Addenda **Arcs** are curves whose endpoints lie in ∂S or on the punctures. We also speak of simple, essential, transversal arcs and of geometric intersection number. The bigon theorem is still true if we add half-bigons that can only be removed by an homotopy that moves the boundary.



§ 3 Change of coordinates principle

How to prove something about an arbitrary collection of s.c.c? The answer: there aren't that many up to the action of $\mathrm{Homeo}^+(S)$.

Given a s.c.c or a simple arc α on S, the cut surface S_α is the surface obtained by cutting along α together with an equivalence relation $\sim_\alpha$ on ∂S_α s.t. $S_\alpha/\sim_\alpha \simeq S$.



The number of connected components of S_α increases by at most 1, depending on whether a tubular neighborhood is connected without α.

Theorem (Change of coordinates) Let α, β be s.c.c. or simple arcs. Then, there is $f \in \mathrm{Homeo}^+(S)$ with $f(\alpha) = \beta$ iff $S_\alpha \simeq S_\beta$.

Proof The only non-trivial thing to prove is that if $S_\alpha \simeq S_\beta$, there is a homeomorphism $\phi: S_\alpha \to S_\beta$ that sends $\sim_\alpha$ to $\sim_\beta$, but this equivalence relation just means mapping a specific boundary circle or arc to another one, easily accomplished by the classification. ∎

Exercise If α is a s.c.c., then $\chi(S_\alpha) = \chi(S)$. If α is a simple arc, then $\chi(S_\alpha) = \chi(S) + 1$.

(Just include α in a cell or simplicial decomposition of S_α & count).

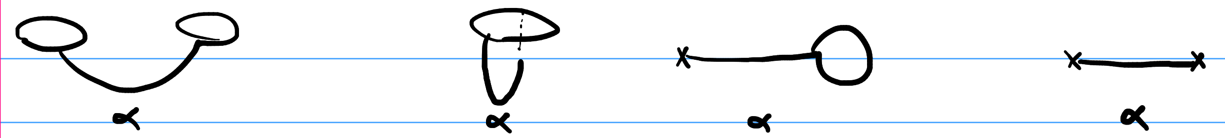
Exercise If α, β are s.c.c. or simple arcs, let $\tilde{\alpha}$ (resp $\tilde{\beta}$) be the image of α (resp β) on S_β (resp S_α). Then $\tilde{\alpha}$ & $\tilde{\beta}$ are disjoint unions of s.c.c and simple arcs, and $(S_\alpha)_{\tilde{\beta}} \simeq (S_\beta)_{\tilde{\alpha}}$. Hence, $(S_\alpha)_\beta = (S_\beta)_\alpha$ makes sense, and we denote it by $S_{\alpha \cup \beta}$.

Let $\alpha_1, \ldots, \alpha_m$ be a collection of s.c.c. and simple arcs. We say it is *separating* if $S_{\alpha_1 \cup \cdots \cup \alpha_m}$ has more connected components than S, and *non-separating* otherwise.

Example 1) If $S = S_{g,b,n}$ and α is a non-separating s.c.c., then $S_\alpha = S_{g-1,b+2,n}$. Hence, any two non-separating s.c.c are equivalent under $\mathrm{Homeo}^+(S, \partial S)$.

examples of non-separating, hence equivalent, curves.

For simple arcs α, there are four equivalence types:



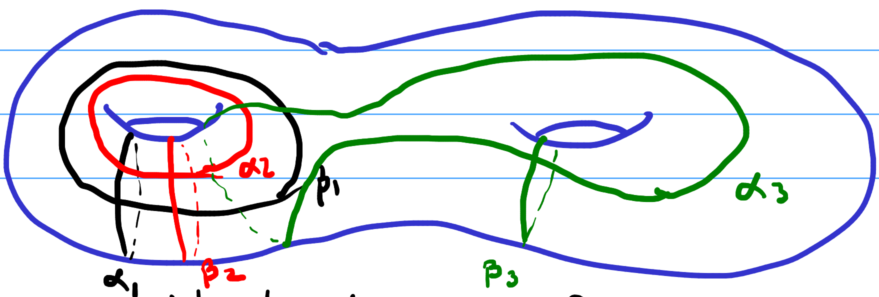
$S_\alpha = S_{g,b-1,m}$ $\quad$ $S_\alpha = S_{g-1,b+1,m}$ $\quad$ $S_\alpha = S_{g,b,m-1}$ $\quad$ $S_\alpha = S_{g,b+1,m-2}$

If we want equivalence under $\mathrm{Homeo}^+(S,\partial S)$, we must also fix the endpoints of α on the boundary.

2) If $S = S_{g,0,0}$ and a s.c.c. α is separating, then $S_\alpha = S_{g-k,0,0} \sqcup S_{k,0,0}$. Call $\min\{g-k,k\}$ the genus of α, it determines α up to $\mathrm{Homeo}^+(S)$.

three genus 1, hence equivalent, separating curves on $S_{3,0,0}$.

3) If $\alpha,\beta \subseteq S_{g,b,m}$ are s.c.c with $|\alpha \cap \beta| = 1$ (in particular, $i(\alpha,\beta)=1$), both are non-separating (α is separating $\Rightarrow [\alpha]=0$ in $H_1(S,\mathbb{Z})$ $\Rightarrow \hat{i}(\alpha,\beta)=0$). We have $S_\alpha = S_{g-1,b+2,m}$ and β becomes a simple arc between two distinct boundary components. Hence, $S_{\alpha\cup\beta} = S_{g-1,b+1,m}$. If α',β' are another such pair, there is $f \in \mathrm{Homeo}^+(S,\partial S)$ such that $f(\alpha)=\alpha'$ and $f(\beta)=\beta'$.



intersection 1 pairs in $S_{2,0,0}$

4) Any finite sequence of disjoint s.c.c $(\alpha_1, \dots, \alpha_m)$ s.t. $\{\alpha_1, \dots, \alpha_m\}$ is non-separating are equivalent under $\mathrm{Homeo}^+(S, \partial S)$.

Exercise* Using that any primitive class in $H_1(S_{g,0,0}, \mathbb{Z})$ has a s.c.c. representative, prove that the image of $\mathrm{Mod}(S_{g,0,0})$ inside $GL_{2g}(\mathbb{Z}) \cong \mathrm{Aut}(H_1(S_{g,0,0}, \mathbb{Z}))$ is the full symplectic group $Sp_{2g}(\mathbb{Z})$. Hint: induction on g.

5) A chain $(\alpha_1, \dots, \alpha_\ell)$ is a sequence of essential s.c.c. such that

$$i(\alpha_i, \alpha_j) = \begin{cases} 1, & \text{if } |i-j| = 1 \\ 0, & \text{if } |i-j| \neq 1 \end{cases}.$$

It is non-separating $\{\alpha_1, \dots, \alpha_\ell\}$ is non-separating. All non-separating chains are equivalent under $\mathrm{Homeo}^+(S)$.

§4 Alexander method

To illustrate the idea behind the Alexander method, let's compute $\mathrm{Mod}(S)$ for all connected S with $\chi(S) \geq 0$. We start with $S = S_{0,1,0} = D^2$, a computation that is key for all that follows.

Lemma (Alexander) $\mathrm{Mod}(D^2)$ is trivial.

Proof Let $f: D^2 \to D^2$ be such that f is the identity on ∂D^2. Think of D^2 as the unit disc on $\mathbb{R}^2$. Define $H: D^2 \times [0,1] \to D^2$ as follows: at time $t \in [0,1)$, do f on the disc of radius t and nothing outside, and $H(-,1) = \mathrm{id}$. Then H is an isotopy. ∎

The same proof works for the punctured disc, putting the puncture on the origin of $\mathbb{R}^2$:

Lemma $\mathrm{Mod}(S_{0,1,1})$ is trivial.

There are two more that are easy to compute:

Proposition $\mathrm{Mod}(S_{0,0,0}) \simeq \mathrm{Mod}(S_{0,0,1}) \simeq \{1\}$. (do this first actually)

Proof Identify $S_{0,0,1} \simeq \mathbb{R}^2$. For $f: \mathbb{R}^2 \to \mathbb{R}^2$ a homeomorphism, we can take $H(x,t) = tx + (1-t)f(x)$. For $f: S^2 \to S^2$, let $\phi: S^2 \to S^2$ be a rotation that maps $f(x_0)$ to x_0 for some $x_0 \in S^2$. Then, $[f] = [\phi \circ f]$ since $[\phi] = [id]$. Since $\phi \circ f$ fixes a point, it induces a map on $S_{0,0,1}$. ∎

We now go to the first non-trivial mapping class group:

Proposition $\mathrm{Mod}(S_{0,0,2}) \simeq \mathbb{Z}/2\mathbb{Z}$, generated by a homeomorphism swapping the punctures.

Proof There is a natural map $\mathrm{Mod}(S_{0,0,2}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ induced by looking at the permutation of the punctures. Any ambient rotation of $\mathbb{R}^3$ that changes them induces a non-trivial element of $\mathrm{Mod}(S_{0,0,2})$ that generates this quotient.

Call those punctures p and q. We claim that if α and β are two simple arcs from p to q, they are isotopic. Since there is some $x_0 \in S_{0,0,2}$ not in $\alpha \cup \beta$, we may think that α and β are arcs in the plane. As in the proof of the bigon theorem, they are either disjoint or contain an inner disc, meaning they are not in minimal position. Hence, we can make α & β

disjoint by an isotopy. Now, $S_\alpha = S_{0,1,0}$ and the image of β is a simple arc connecting two distinct points on the boundary. Hence, the image of β is separating by the Jordan curve theorem and a calculation with Euler characteristics shows $S_{\alpha \cup \beta} = S_{0,1,0} \sqcup S_{0,1,0}$, that is, $\alpha \cup \beta$ bounds two embedded discs. Hence, they are isotopic.

Now take $f \in \mathrm{Mod}(S_{0,0,2})$ and assume that $f(p)=p$, $f(q)=q$. Then, $f(\alpha)$ is isotopic to α. It is a standard result on differential geometry that an isotopy of curves can be extended to an isotopy on S (see, for example, Lee). Hence, $[f]=[g]$ where g fixes α point-wise.

Then, g restricts to $S_\alpha = S_{0,1,0} \simeq D^2$. Since $\mathrm{Mod}(D^2) = \{1\}$, $[g]$ is trivial. ∎

What was the idea: find a homotopy class fixed by f, use the extension of isotopies and cut along α to get a disk. Let's see another example:

Proposition $\mathrm{Mod}(S_{0,2,0}) \simeq \mathbb{Z}$.

My proof We work in the differential category, and think of $S_{0,2,0} \hookrightarrow \mathbb{R}^2$ as $\{(x,y) \in \mathbb{R}^2 \mid 1 \leq \sqrt{x^2+y^2} \leq 2\}$. Let δ be the arc given by the straight line between $(0,2)$ and $(0,1)$. Divide $S_{0,2,0}$ in two annuli A_- and A_+, from $y \in [1, \frac{3}{2}]$ and $y \in [\frac{3}{2}, 2]$, so that it naturally divides δ into δ^+ and δ^-.

Given $f \in \mathrm{Diff}^+(S, \partial S)$, let f^+ and f^- be the new maps given by applying f in A_- & A_+ and the identity elsewhere. It is clear that $[f]=[f^+]=[f^-]$ by contracting either A_+ or A_-. Recall that if γ is an arc in $\mathbb{R}^2 \setminus \{(0,0)\}$, its winding number around the origin is

$$\mathrm{ind}(\gamma) = \frac{1}{2\pi}\int_\gamma \frac{x\,dy - y\,dx}{x^2+y^2}$$

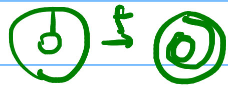
We have that $\mathrm{ind}(\gamma) \in \mathbb{Z}$ and that it only depends on $[\gamma]$.
Hence, $f \longmapsto \mathrm{ind}([\delta]-[f\delta])$
$\mathrm{Mod}(S)$ to $\mathbb{Z}$.

To see that it is an homomorphism, notice that $[f\circ g] = [f^{+}\circ g^{-}]$ and thus

$$\mathrm{ind}([(f\circ g)\delta]) = \mathrm{ind}([f^{+}\delta]+[g^{-}\delta]) = \mathrm{ind}([f^{+}\delta]) + \mathrm{ind}([g^{-}\delta])$$
$$= \mathrm{ind}([f\delta]) + \mathrm{ind}([g\delta]).$$

It is also surjective: taking

$$f(x,y) = \begin{pmatrix} \cos 2\pi(\sqrt{x^2+y^2}-1) & \operatorname{sen} 2\pi(\sqrt{x^2+y^2}-1) \\ -\operatorname{sen} 2\pi(\sqrt{x^2+y^2}-1) & \cos 2\pi(\sqrt{x^2+y^2}-1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$



we have $\mathrm{ind}([\delta]-[f\delta]) = 1$.

Lastly, assume that $f \in \mathrm{Diff}^{+}(S,\partial S)$ is such that $\mathrm{ind}([\delta]-[f\delta])$ vanishes. Since ind generates $H^1_{dR}(\mathbb{R}^2\setminus\{0\})$, we have that $[\delta]-[f\delta]$ vanishes in the homology of the punctured plane and thus bounds a disc as $H_1(\mathbb{R}^2\setminus\{0\}) \cong \pi_1(\mathbb{R}^2\setminus\{0\})$.

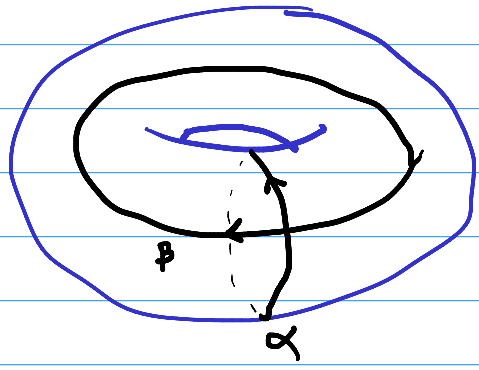
This shows $(\delta)=(f\delta)$, and hence by the extension of isotopies $[f]=[g]$ where g fixes δ pointwise. Hence, g induces an element of $\mathrm{Mod}(S_\delta) = \mathrm{Mod}(S_{0,1,0}) = \{1\}$. ∎

Spoiler The generator of $\mathrm{Mod}(S_{0,2,0})$ is called a Dehn twist. Since the tubular neighborhood of any s.c.c. on a surface is homeomorphic to $S_{0,2,0}$, we can perform Dehn twists on any surface. Sergio will talk more about this.

We keep going:

Proposition $\mathrm{Mod}(S_{1,0,0}) \cong SL_2\mathbb{Z}$

Proof Consider the following generators of $\pi_1(S_{1,0,0}) \cong H_1(S_{1,0,0})$



The action of $\mathrm{Mod}(S)$ on $H_1(S) \cong \mathbb{Z}^2$ gives us a map $\mathrm{Mod}(S) \xrightarrow{\psi} GL_2\mathbb{Z}$. It is not hard to show that

$$1 = \hat{i}(\alpha,\beta) = \hat{i}(f\alpha, f\beta) = \det(\psi f).$$

Hence, the image lies in $SL_2\mathbb{Z}$. Moreover, any matrix $A \in SL_2\mathbb{Z}$ defines an orientation preserving map $\mathbb{R}^2 \to \mathbb{R}^2$ that preserves the lattice $\mathbb{Z}^2$, and hence induces $f_A \in \mathrm{Homeo}^+(S)$. It is a direct computation on the lifts of α and β to see that $\psi(f_A) = A$. This shows the map $\mathrm{Mod}(S) \xrightarrow{\psi} SL_2\mathbb{Z}$ is surjective.

If $\psi(f) = \mathrm{Id}_2$, then $(\alpha) = (f\alpha)$, so by the extension of isotopies there is $g \in \mathrm{Homeo}^+(S)$ fixing α pointwise such that $[g]=[f]$. However, cutting along α yields $S_\alpha \cong S_{0,2,0}$ and β becomes the arc δ of the previous proof. Since $(\beta) = (f\beta) = (g\beta)$, we have $(g\delta) = (\delta)$ and thus we find $h \in \mathrm{Homeo}^+(S_{0,2,0})$ such that $(h) = (g)$ and h fixes δ pointwise. Cutting δ yields a disk and finishes the proof. ∎

Exercise Show $\mathrm{Mod}(S_{1,0,1}) \cong SL_2\mathbb{Z}$.

We are now ready to state our main theorem, the Alexander method. We say a family $\alpha_1, \dots, \alpha_k$ of s.c.c. and simple arcs **fills** S if $S_{\alpha_1,\dots,\alpha_k}$ is a union of disks and 1-punctured disks.

Theorem Let $f \in \text{Homeo}^+(S, \partial S)$ and $\gamma_1, \dots, \gamma_m$ be a family of essential simple closed curves & arcs such that:

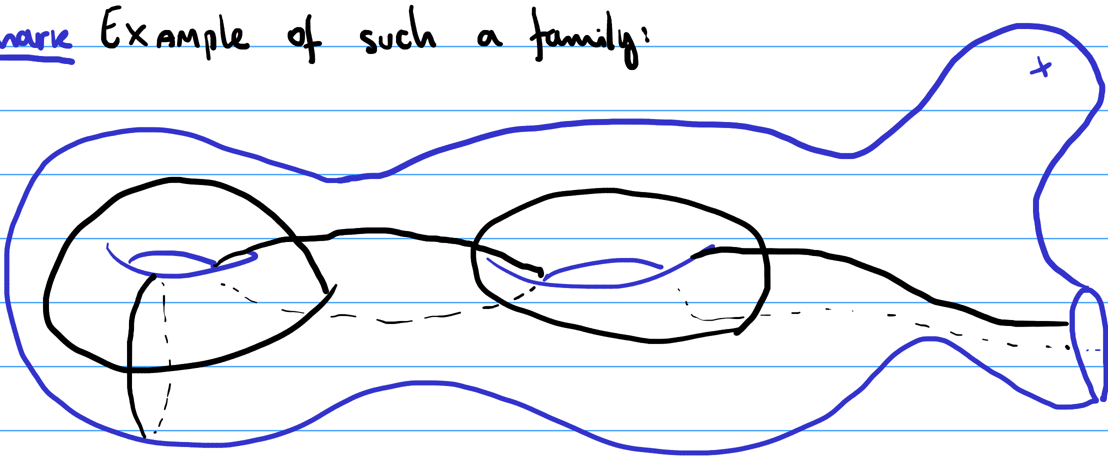
1. γ_i are pairwise in minimal position.
2. γ_i are pairwise non-isotopic
3. For i,j,k distinct, one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, $\gamma_j \cap \gamma_k$ is empty.

(1) If there is $\sigma \in \text{Sym}(m)$ s.t. $f(\gamma_i)$ is isotopic to $\gamma_{\sigma i}$ relative to ∂S for each i, then $f(\cup \gamma_i)$ is isotopic to $\cup \gamma_i$ relative to ∂S.

If we regard $\cup \gamma_i$ as a graph, then this isotopy gives an automorphism of this graph.

(2) If $\gamma_1, \dots, \gamma_m$ fills S and f is the identity on $\cup \gamma_i$, then $[f]=1$. Otherwise, $[f]$ has finite order in $\text{Mod}(S)$.

Remark Example of such a family:



Since the curves have algebraic intersection ≤ 1, they are in minimal position. Also clearly pairwise non-isotopic since they are distinct in $H_1(S_{2,1,1}, \mathbb{Z})$. No three also intersect pairwise. To see that they fill S, first observe that the resulting cut surface has 2 connected components of 1 boundary each, one punctured and one not, and the total Euler characteristic sums to 1. Hence, it must be $S_{0,1,1} \sqcup S_{0,1,0}$.